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On the Spectrum of Generalized Zero-divisor Graph of the Ring $\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}$

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Abstract

The generalized zero-divisor graph of a commutative ring R denoted by $\Gamma'(R)$, is a simple (undirected) graph with the vertex set consisting of all nonzero zero-divisors in R and two vertices x and y are adjacent if $x^n y = 0$ or $y^n x = 0$ for some positive integer n. For the distinct primes p, q, r and positive integers k_1, k_2, k_3, we determine the adjacency matrix and eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$. Also, we calculate the clique number, diameter, girth and stability number of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ and verify Beck's conjecture for $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

Keywords: generalized zero-divisor graph; eigenvalues; adjacency matrix.

1 Introduction

Let G be a simple graph with vertex set V and edge set E. The length of the shortest path between two vertices x and y is called the distance between x and y and is denoted by $d(x, y)$. The diameter of G is the maximum of the distances between the vertices and is denoted as $\text{diam}(G)$. The girth of G denoted by $\text{gr}(G)$, is the length of the shortest cycle in G. The girth of G is defined to be ∞ if G contains no cycles. The clique number of G is the number of vertices in its largest clique and is denoted by $\omega(G)$. The cardinality of the largest set of pairwise non-adjacent vertices in G is called the stability number, $\alpha(G)$. For a graph G with n vertices, $A(G) = [a_{ij}]_{n\times n}$ denotes the adjacency matrix of G, where $a_{ij} = 1$ if there is an edge between i^{th} and j^{th} vertices of G and 0 otherwise. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $A(G)$. The adjacency spectrum of G is the multiset, denoted by $\sigma_A(G) = \left\{\mu_1^{(s_1)}, \ldots, \mu_n^{(s_n)}\right\}$. A mapping $*$ on an associative ring is called an involution if for all $x, y \in R : (x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$. A ring equipped with involution $*$ is called a $*$-ring.

For a commutative ring, Beck [3] introduced the concept of a zero-divisor graph. Building on this idea, Anderson and Livingston [1] introduced the zero-divisor graph for commutative rings. Redmond [14] extended this idea of zero-divisor graph to non-commutative rings. Later, Patil and Waphare [9] extended this idea of zero-divisor graphs to rings with involution. More recently, Lande and Khairnar [5] extended the concept of zero-divisor graph to the generalized zero-divisor graph. For a $*$-ring R, they associated a simple (undirected) graph denoted by $\Gamma'(R)$, with the vertex set $Z^*(R)$ and two vertices x and y are adjacent if $x^n y^* = 0$ or $y^n x^* = 0$ for some positive integer n [5].

Spectral graph theory and zero-divisor graphs serve as bridges between abstract algebra, combinatorics and applied fields, enriching both theoretical understanding and practical design in disciplines such as coding theory, network science and algebraic systems classification. Their interplay with algebraic structures allows for deeper analysis, optimal design and innovative applications in both pure and applied mathematics. Many authors have done the study of spectral properties of the zero-divisor graph. Magi et al. [7] determined the eigenvalues of $A\left(\Gamma\left(\mathbb{Z}_{p^2q^2}\right)\right)$ for primes p, q. They also computed the diameter, stability number, girth and clique number of $\Gamma\left(\mathbb{Z}_{p^2q^2}\right)$. Mönius [8] investigated the eigenvalues of $\Gamma'(R)$ for a finite commutative ring R. Pirzada et al. [13] studied the spectrum of the zero-divisor graph $\Gamma\left(\mathbb{Z}_{p^Mq^N}\right)$ for primes p, q and integers $M, N > 0$. Pirzada et al. [10] analyzed the structure of $\Gamma\left(\mathbb{Z}_n\right)$ for $n = p^{N_1}q^{N_2}r$, for primes $2 < p < q < r$ and integers $N_1, N_2 > 0$. Pirzada et al. [11] studied signless Laplacian eigenvalues of the zero-divisor graph of the ring $\mathbb{Z}_{p^Mq^N}$. The study of signless Laplacian spectrum of the zero divisor graphs of the ring $\mathbb{Z}_n$ is explored in [12].

Recently, Rehman et al. [15] studied properties of the signless Laplacian spectrum of weakly zero-divisor graph of commutative ring $\mathbb{Z}_n$. Furthermore, Rehman et al. [16] studied normalized distance Laplacian eigenvalues of the zero-divisor graph $\mathbb{Z}_n$. Ashraf et al. [2] studied A_α spectrum of the zero-divisor graph of the ring $\mathbb{Z}_n$. Additionally, Semil@Ismail et al. [17] explored distance based topological indices of the zero divisor graph for some commutative rings. Zai et al. [19] determined the non-zero divisor graph of rings of integers modulo n, where $n = 8k$ and $k \leq 3$. Lande and Khairnar [6] determined the adjacency spectrum of the generalized zero-divisor graph of the ring $\mathbb{Z}_{p^\alpha q^\beta}$, where p, q are distinct primes and α, β are positive integers.

In this paper, we extend this study and determine the eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ for distinct primes p, q, r and integers $k_1, k_2, k_3 > 0$. In Section 2, we determine eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$. In Section 3, we compute the clique number, diameter, girth and stability number of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

Furthermore, we prove Beck's conjecture for $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

2 Eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$

In this section, we begin by recalling some results that will be used in the subsequent analysis. We then determine the eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$, where p, q, r are primes and k_1, k_2, k_3 are positive integers.

The number of positive integers that are less than n and that are relatively prime to n is called $\phi(n)$, Euler's totient function. We first summarize well-known results.

Lemma 2.1. *Let ϕ be Euler's totient function. Then, $|Z^*(\mathbb{Z}_n)| = n - 1 - \phi(n)$.*

The greatest common divisor of x and n denoted by (x, n) and let $T_d = \{x \in \mathbb{Z}_n : (x, n) = d\}$. The cardinality of T_d denoted by $|T_d|$ is given in the result below;

Proposition 2.1. [18] *Suppose d is a divisor of n. Then, $|T_d| = \phi\left(\frac{n}{d}\right)$.*

For an integer $n > 1$, the canonical decomposition is given by $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ for primes $p_1, p_2, \dots, p_r$ and positive integers $k_1, k_2, \dots, k_r$.
We know that, $\phi(n) = n\left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 1}{p_2}\right)\cdots\left(\frac{p_r - 1}{p_r}\right)$.

Definition 2.1. [4] *Let $\{1, 2, \dots, n\}$ be the n vertices of graph K and let $G_1, G_2, \dots, G_n$ be graphs. Then, the K–generalized join of the graphs $G_1, G_2, \dots, G_n$, is denoted by $\bigvee_K \{G_1, G_2, \dots, G_n\}$ and it is a graph obtained by replacing each vertex i of K by the graph G_i and there is an edge between any two vertices of G_i and G_j if and only if there is an edge between the vertices i and j in K.*

Theorem 2.1. [4] *Let G_i be n pairwise disjoint r_i–regular graphs of order n_i and G be a graph with vertices $\{1, 2, \dots, n\}$. Then, the adjacency spectrum of $G = \bigvee\{G_1, G_2, \dots, G_n\}$ is given by,*

$$\sigma_A(G) = \left(\bigcup_{i=1}^{n} \left(\sigma_A(G_i) \setminus \{r_i\}\right)\right) \bigcup \sigma\left(C_A(G)\right),$$

where

$$C_A(G) = (c_{ij})_{n \times n} = \begin{cases} r_i, & i = j, \\ \sqrt{n_i n_j}, & ij \in E(G), \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Definition 2.2. [5, Definition 2.1] *Let R be a $*$-ring, the generalized zero-divisor graph denoted by $\Gamma'(R)$ is a simple (undirected) graph with vertex set $Z^*(R)$ and two distinct vertices x and y are adjacent if and only if $x^n y^* = 0$ or $y^n x^* = 0$, for some positive integer n.*

For a commutative ring R, the involution $*$ is an identity involution. Hence there is an edge between x and y if $x^n y = 0$ or $y^n x = 0$ for some positive integer n.

Some notations:

$$\begin{aligned} 0_{n_i,n_j} &: \text{A matrix of all zeros of order } n_i \times n_j, \\ 1_{n_i,n_j} &: \text{A matrix of ones of order } n_i \times n_j, \\ I &: \text{An identity matrix.} \end{aligned}$$

If each element in set X is adjacent to each element of set Y, we denote it as $X \sim Y$. If no element of set X is adjacent to any elements of set Y, we denote it as $X \nsim Y$.

The following result determines eigenvalues of $\Gamma'(\mathbb{Z}_{prq})$.

Theorem 2.2. *Let p, r, q be primes and let the sets $X_1, X_2, \ldots, X_6$ be as defined in (4) with $n_i = |X_i|$,*

(a) *The adjacency matrix of $\Gamma'(\mathbb{Z}_{prq})$ is*

$$A(\Gamma'(\mathbb{Z}_{prq})) = \left[\begin{array}{c|c} 0_{n_1+n_2+n_3,n_1+n_2+n_3} & B_{n_1+n_2+n_3,n_4+n_5+n_6} \\ \hline B^t & (1-I)_{n_4+n_5+n_6,n_4+n_5+n_6} \end{array}\right], \tag{2}$$

where

$$B_{n_1+n_2+n_3,n_4+n_5+n_6} = \begin{bmatrix} 0_{n_1,n_4} & 0_{n_1,n_5} & 1_{n_1,n_6} \\ 0_{n_2,n_4} & 1_{n_2,n_5} & 0_{n_2,n_6} \\ 1_{n_3,n_4} & 0_{n_3,n_5} & 0_{n_3,n_6} \end{bmatrix}.$$

(b) *Zero is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{prq}))$ with multiplicity $pq + qr + pr - (p + r + q) - 6$.*

(c) *The eigenvalues of the matrix M, where,*

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{n_1 n_6} \\ 0 & 0 & 0 & 0 & \sqrt{n_2 n_5} & 0 \\ 0 & 0 & 0 & \sqrt{n_3 n_4} & 0 & 0 \\ 0 & 0 & \sqrt{n_4 n_3} & 0 & \sqrt{n_4 n_5} & \sqrt{n_4 n_6} \\ 0 & \sqrt{n_5 n_2} & 0 & \sqrt{n_5 n_4} & 0 & \sqrt{n_5 n_6} \\ \sqrt{n_6 n_1} & 0 & 0 & \sqrt{n_6 n_4} & \sqrt{n_6 n_5} & 0 \end{bmatrix}, \tag{3}$$

gives the remaining six eigenvalues of $A(\Gamma'(\mathbb{Z}_{prq}))$.

Proof.

(a) Let p, r, q be primes and $n = prq$. Consider a partition of $V(\Gamma'(\mathbb{Z}_{prq}))$ into six subsets based on the zero-divisors of $\mathbb{Z}_n$. Define:

$$\begin{aligned} X_1 &= \{x \in \mathbb{Z}_n : (x, n) = p\}, \\ X_2 &= \{x \in \mathbb{Z}_n : (x, n) = q\}, \\ X_3 &= \{x \in \mathbb{Z}_n : (x, n) = r\}, \\ X_4 &= \{x \in \mathbb{Z}_n : (x, n) = pq\}, \\ X_5 &= \{x \in \mathbb{Z}_n : (x, n) = pr\}, \\ X_6 &= \{x \in \mathbb{Z}_n : (x, n) = qr\}. \end{aligned} \tag{4}$$

All the sets $X_1, X_2, \ldots, X_6$ are mutually disjoint, forming a partition,

$$P = \{X_1, X_2, \ldots, X_6\}, \tag{5}$$

of $V\left(\Gamma'\left(\mathbb{Z}_{prq}\right)\right)$.

Next, we find the cardinality of the sets $X_1, X_2, \ldots, X_6$. Let $x \in X_1$. Then, $(x, n) = p$. Therefore,

$$n_1 = |X_1| = \phi\left(\frac{prq}{p}\right) = (r-1)(q-1).$$

Similarly,

$$\begin{aligned} n_2 &= |X_2| = (r-1)(p-1), \\ n_3 &= |X_3| = (q-1)(p-1), \\ n_4 &= |X_4| = r-1, \\ n_5 &= |X_5| = q-1, \\ n_6 &= |X_6| = p-1. \end{aligned}$$

Since $\phi(prq) = (r-1)(q-1)(p-1)$, the number of nonzero zero-divisors in $\mathbb{Z}_{prq}$ is

$$\begin{aligned} n - 1 - \phi(n) &= prq - (r-1)(q-1)(p-1) - 1 \\ &= qr + pr + pq - (r + q + p). \end{aligned} \tag{6}$$

The order of matrix $A\left(\Gamma'\left(\mathbb{Z}_{prq}\right)\right)$ is

$$\left|A\left(\Gamma'\left(\mathbb{Z}_{prq}\right)\right)\right| = \sum_{i=1}^{6} |X_i| = qr + pr + pq - (r + q + p),$$

which is consistent with the result from (6).

Next, we determine the neighborhoods of the elements in $\Gamma'\left(\mathbb{Z}_{prq}\right)$:

(a) Let $a \in X_1,\ b \in X_6$. Then, $ab = 0$.
Furthermore, for any $b \in X_1 \cup X_2 \cup \ldots \cup X_5$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Thus, every vertex in X_1 is adjacent to every vertex in X_6 so $X_1 \sim X_6$. No vertex of X_1 is adjacent to any vertex in $X_1, X_2, \ldots, X_5$, so $X_1 \nsim X_i$ for $i = 1, 2, \ldots, 5$.

(b) Let $a \in X_2, b \in X_5$. Then, $ab = 0$.
Furthermore, for any $b \in X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_6$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Thus, every vertex in X_2 is adjacent to every vertex in X_5, so $X_1 \sim X_5$. None of the vertex of X_2 is adjacent to any of the vertex of $X_1, \ldots, X_4$ and X_6, so $X_1 \nsim X_i$ for $i = 1, 2, \ldots, 4, 6$.

(c) Let $a \in X_3, b \in X_4$. Then, $ab = 0$.
Also, for any $b \in X_1 \cup X_2 \cup X_3 \cup X_5 \cup X_6$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, every vertex in X_3 is adjacent to every vertex in X_4, so $X_3 \sim X_4$. None of the vertex in X_3 is adjacent to any of the vertex in X_1, X_2, X_3, X_5 and X_6, so $X_3 \nsim X_i$ for $i = 1, 2, 3, 5, 6$.

(d) Let $a \in X_4, b \in X_3, X_5, X_6$. Then, $ab = 0$.
And, $a^k b \neq 0$ and $b^k a \neq 0$ for any $b \in X_1 \cup X_2 \cup X_4$ and for any positive integer k. Therefore, $X_4 \sim X_3$, $X_4 \sim X_5$, $X_4 \sim X_6$ and $X_4 \nsim X_1$, $X_4 \nsim X_2$, $X_4 \nsim X_4$.

(e) Let $a \in X_5, b \in X_2, X_4, X_6$. Then, $ab = 0$.
Also, for any $b \in X_1 \cup X_3 \cup X_5$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, $X_5 \sim X_2$, $X_5 \sim X_4$, $X_5 \sim X_6$ and $X_5 \nsim X_1$, $X_5 \nsim X_3$, $X_5 \nsim X_5$.

(f) Let $a \in X_6, b \in X_1, X_4, X_5$. Then, $ab = 0$.
Also, for any $b \in X_2 \cup X_3 \cup X_6$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, $X_6 \sim X_1$, $X_6 \sim X_4$, $X_6 \sim X_5$ and $X_6 \nsim X_2$, $X_6 \nsim X_3$, $X_6 \nsim X_6$.

Since there is an edge between the vertices of X_1 to the vertices of X_6, we obtain a block of ones in the adjacency matrix corresponding to the row for X_1 and the column for X_6. Similarly, blocks of zeros and ones appear for the remaining vertices, based on their adjacency relationships.

Thus, the adjacency matrix of $\Gamma'(\mathbb{Z}_{prq})$ with row and column headings $X_1, X_2, \ldots, X_6$ is given by,

$$A(\Gamma'(\mathbb{Z}_{prq})) = \begin{bmatrix} 0_{n_1,n_1} & 0_{n_1,n_2} & 0_{n_1,n_3} & 0_{n_1,n_4} & 0_{n_1,n_5} & 1_{n_1,n_6} \\ 0_{n_2,n_1} & 0_{n_2,n_2} & 0_{n_2,n_3} & 0_{n_2,n_4} & 1_{n_2,n_5} & 0_{n_2,n_6} \\ 0_{n_3,n_1} & 0_{n_3,n_2} & 0_{n_3,n_3} & 1_{n_3,n_4} & 0_{n_3,n_5} & 0_{n_3,n_6} \\ 0_{n_4,n_1} & 0_{n_4,n_2} & 1_{n_4,n_3} & 0_{n_4,n_4} & 1_{n_4,n_5} & 1_{n_4,n_6} \\ 0_{n_5,n_1} & 1_{n_5,n_2} & 0_{n_5,n_3} & 1_{n_5,n_4} & 0_{n_5,n_5} & 1_{n_5,n_6} \\ 1_{n_6,n_1} & 0_{n_6,n_2} & 0_{n_6,n_3} & 1_{n_6,n_4} & 1_{n_6,n_5} & 0_{n_6,n_6} \end{bmatrix}, \tag{7}$$

where $0_{n_i,n_j}$ is a matrix of all zeros of order $n_i \times n_j$, $1_{n_i,n_j}$ is a matrix of all ones of order $n_i \times n_j$. Let B be a matrix,

$$B_{n_1+n_2+n_3,n_4+n_5+n_6} = \begin{bmatrix} 0_{n_1,n_4} & 0_{n_1,n_5} & 1_{n_1,n_6} \\ 0_{n_2,n_4} & 1_{n_2,n_5} & 0_{n_2,n_6} \\ 1_{n_3,n_4} & 0_{n_3,n_5} & 0_{n_3,n_6} \end{bmatrix}.$$

Thus,

$$A(\Gamma'(\mathbb{Z}_{prq})) = \left[\begin{array}{c|c} 0_{n_1+n_2+n_3,n_1+n_2+n_3} & B \\ \hline B^t & (1-I)_{n_4+n_5+n_6,n_4+n_5+n_6} \end{array}\right].$$

(b) The adjacency matrix $A(\Gamma'(\mathbb{Z}_{prq}))$ is given in (7). Since $A(\Gamma'(\mathbb{Z}_{prq}))$ is a real and symmetric matrix, for all the eigenvalues the algebraic multiplicities and the geometric multiplicities are the same. Note that,

$$\det(A(\Gamma'(\mathbb{Z}_{prq}))) = 0,$$

which implies that 0 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{prq}))$. The nullity of $A(\Gamma'(\mathbb{Z}_{prq}))$ gives the geometric multiplicity of the eigenvalue 0. By performing elementary row transformations on $A(\Gamma'(\mathbb{Z}_{prq}))$, in the transformed matrix we find that the number of zero rows is

$$\sum_{i=1}^{6} |X_i| - 6 = pq + qr + pr - (p + r + q) - 6.$$

Thus, the nullity of $A(\Gamma'(\mathbb{Z}_{prq}))$ is $pq + qr + pr - (p + r + q) - 6$. Therefore, the eigenvalue 0 has the multiplicity $pq + qr + pr - (p + r + q) - 6$.

(c) The graph $\Gamma'(\mathbb{Z}_{prq})$ is depicted in the Figure 1, based on the structure described in (7).

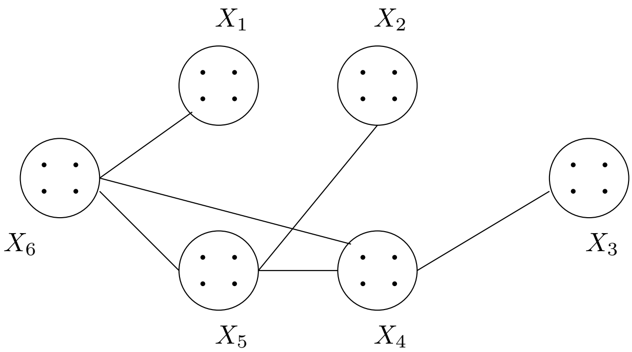

Figure 1: The generalized zero-divisor graph $\Gamma'(\mathbb{Z}_{prq})$.

Since, the order of the matrix $A(\Gamma'(\mathbb{Z}_{prq}))$ is $qr+pr+pq-(r+q+p)$ and 0 is the eigenvalue with multiplicity $qr+pr+pq-(r+q+p)-6$, it follows that there are 6 remaining eigenvalues of $A(\Gamma'(\mathbb{Z}_{prq}))$. Since the trace of the matrix equals the sum of all eigenvalues, the remaining 6 eigenvalues must have sum 0. Using the observations above, Theorem 2.1, (1) and the matrix $A(\Gamma'(\mathbb{Z}_{prq}))$ as given in (7) we get that the remaining 6 eigenvalues of $A(\Gamma'(\mathbb{Z}_{prq}))$ are the eigenvalues of the matrix M where,

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{n_1 n_6} \\ 0 & 0 & 0 & 0 & \sqrt{n_2 n_5} & 0 \\ 0 & 0 & 0 & \sqrt{n_3 n_4} & 0 & 0 \\ 0 & 0 & \sqrt{n_4 n_3} & 0 & \sqrt{n_4 n_5} & \sqrt{n_4 n_6} \\ 0 & \sqrt{n_5 n_2} & 0 & \sqrt{n_5 n_4} & 0 & \sqrt{n_5 n_6} \\ \sqrt{n_6 n_1} & 0 & 0 & \sqrt{n_6 n_4} & \sqrt{n_6 n_5} & 0 \end{bmatrix}.$$

□

Next, we determine the eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)$, for distinct primes p, r, q and integer $k_1 > 1$.

Theorem 2.3. *Let $k_1 > 1$ be an integer and p, q, r distinct primes. Define the sets $X_1, X_2, \ldots, X_7$ as given in (10) and let $n_i = |X_i|$;*

(*a*) *The adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)$ is*

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right) = \left[\begin{array}{c|c|c} 0_{n_1+n_2+n_3,n_1+n_2+n_3} & B_{n_1+n_2+n_3,n_4+n_5+n_6} & 1_{n_1+n_2+n_3,n_7} \\ \hline B^t & (1-I)_{n_4+n_5+n_6,n_4+n_5+n_6} & 1_{n_4+n_5+n_6,n_7} \\ \hline 1^t_{n_1+n_2+n_3,n_7} & 1^t_{n_4+n_5+n_6,n_7} & (1-I)_{n_7,n_7} \end{array}\right], \tag{8}$$

where B is the matrix,

$$B = \begin{bmatrix} 0_{n_1,n_4} & 0_{n_1,n_5} & 1_{n_1,n_6} \\ 0_{n_2,n_4} & 1_{n_2,n_5} & 0_{n_2,n_6} \\ 1_{n_3,n_4} & 0_{n_3,n_5} & 0_{n_3,n_6} \end{bmatrix}.$$

(*b*) *Zero is an eigenvalue of* $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$ *with multiplicity,*

$$p^{k_1-1}\left[pq+qr+pr-(p+r+q)\right]-6,$$

and -1 *is an eigenvalue with multiplicity* $p^{k_1-1}-2$.

(*c*) *The remaining* 7 *eigenvalues of* $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$ *are the eigenvalues of* $M=[m_{ij}]_{7\times 7}$ *where,*

$$m_{ij}=\begin{cases} n_7-1, & i=j=7,\\ \sqrt{n_in_j}, & X_i\sim X_j,\\ 0, & \textit{otherwise}, \end{cases} \tag{9}$$

$n_i=|X_i|$.

Proof.

(a) Let $n=p^{k_1}qr$, where p,q,r be distinct primes and $k_1>1$ be an integer. Suppose $V\left(\Gamma'\left(\mathbb{Z}_n\right)\right)$ is partitioned into the following classes. Let

$$\begin{aligned} X_1&=\{x\in\mathbb{Z}_n\ :\ (x,n)=p^i, \quad i=1,2,\ldots,k_1\},\\ X_2&=\{x\in\mathbb{Z}_n\ :\ (x,n)=q\},\\ X_3&=\{x\in\mathbb{Z}_n\ :\ (x,n)=r\},\\ X_4&=\{x\in\mathbb{Z}_n\ :\ (x,n)=p^iq, \quad i=1,2,\ldots,k_1\},\\ X_5&=\{x\in\mathbb{Z}_n\ :\ (x,n)=p^ir, \quad i=1,2,\ldots,k_1\},\\ X_6&=\{x\in\mathbb{Z}_n\ :\ (x,n)=qr\},\\ X_7&=\{x\in\mathbb{Z}_n\ :\ x=kprq, \quad k=1,2,\ldots,p^{k_1-1}-1\}. \end{aligned} \tag{10}$$

Observe that, X_7 contains the nilpotent elements in $\mathbb{Z}_{p^{k_1}qr}$ and all the sets $X_1,X_2,\ldots,X_7$ are mutually disjoint. Hence,

$$\{X_1\cup X_2\cup\ldots\cup X_7\}, \tag{11}$$

form a partition of $V\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$.

Next, we find the cardinality of the sets $X_1,X_2,\ldots,X_7$. Let $x\in X_1$. Then, for some i, $(x,n)=p^i$. Therefore,

$$\begin{aligned} n_1=|X_1|&=\phi(p^{k_1-1}qr)+\phi(p^{k_2-1}qr)+\cdots+\phi(qr)\\ &=p^{k_1-2}(r-1)(q-1)(p-1)+\cdots+(r-1)(q-1)(p-1)+(r-1)(q-1)\\ &=p^{k_1-1}(r-1)(q-1). \end{aligned}$$

Similarly,

$$\begin{aligned} n_2&=|X_2|=p^{k_1-1}(r-1)(p-1),\\ n_3&=|X_3|=p^{k_1-1}(p-1)(q-1),\\ n_4&=|X_4|=\phi\left(\frac{n}{pq}\right)+\phi\left(\frac{n}{p^2q}\right)+\cdots+\phi\left(\frac{n}{p^{k_1}q}\right)=p^{k_1-1}(r-1),\\ n_5&=|X_5|=\phi\left(\frac{n}{pr}\right)+\phi\left(\frac{n}{p^2r}\right)+\cdots+\phi\left(\frac{n}{p^{k_1}r}\right)=p^{k_1-1}(q-1),\\ n_6&=|X_6|=p^{k_1-1}(p-1),\\ n_7&=|X_7|=p^{k_1-1}-1. \end{aligned}$$

The order of the matrix $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$ is

$$\begin{aligned}\sum_{i=1}^{7}|X_i| &= p^{k_1-1}(r-1)(q-1)+p^{k_1-1}(r-1)(p-1)+p^{k_1-1}(q-1)(p-1)\\ &\quad + p^{k_1-1}(r-1)+p^{k_1-1}(q-1)+p^{k_1-1}(p-1)+p-1\\ &= p^{k_1-1}(pq+pr+qr-(p+r+q)+1)-1. \end{aligned} \tag{12}$$

$\phi(n) = \phi(p^{k_1}qr) = (r-1)(q-1)p^{k_1-1}(p-1)$. The cardinality of zero-divisors in $\mathbb{Z}_{p^{k_1}qr}$ is given by $n-\phi(n)-1 = p^{k_1}qr-(r-1)(q-1)p^{k_1-1}(p-1)-1$, which simplifies to $p^{k_1-1}(pr+qr+pq-(r+q+p)+1)-1$, as shown in (12).

Next, we determine the neighborhoods of the vertices.

(a) Since X_7 contains the nilpotent elements, each element in X_7 is adjacent to all the remaining vertices, that is $X_7 \sim X_i$, for $i=1,2,\ldots,7$.

(b) Let $a\in X_1$, $b\in X_6$. Then, $a^m b=0$ for some $m>0$. Also, for any $b\in V\left(\Gamma'\left(R\right)\right)\setminus\{X_6\cup X_7\}$ and for any k, $a^k b\neq 0$ and $b^k a\neq 0$. Therefore every element in X_1 is adjacent to every element in X_6, X_7. Thus, $X_1\sim X_6, X_7$. None of the elements in X_1 is adjacent to elements in $V\left(\Gamma'\left(R\right)\right)\setminus\{X_6\cup X_7\}$.

(c) Similarly, we get

$$\begin{array}{llll} X_2\sim X_5, & X_2\sim X_7, & & \\ X_3\sim X_4, & X_3\sim X_7, & & \\ X_4\sim X_3, & X_4\sim X_5, & X_4\sim X_6, & X_4\sim X_7,\\ X_5\sim X_2, & X_5\sim X_4, & X_5\sim X_6, & X_5\sim X_7,\\ X_6\sim X_1, & X_6\sim X_4, & X_6\sim X_5, & X_6\sim X_7,\\ X_7\sim X_1,\ldots,X_7. & & & \end{array}$$

Since each vertex in X_1 is adjacent to each vertex in X_6, we obtain a block of ones of size $n_1\times n_6$ corresponding to the row X_1 and the column X_6. Additionally, no vertex in X_1 is adjacent to any vertex in X_2, resulting in a block of zeros of size $n_1\times n_2$ corresponding to row X_1 and column X_2.

Similarly, we can construct blocks of zeros and ones for the remaining vertex sets. Note that, for nilpotent elements, the entries on the diagonal are considered zero. Therefore, for row X_7 and column X_7 we obtain a block of $1-I$ of size $n_7\times n_7$. In the same manner, the adjacency relationships between the other vertices can also be determined. Hence, the adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)$ with row and column headings $X_1, X_2, \ldots, X_7$ is given by,

$$\begin{bmatrix} 0_{n_1,n_1} & 0_{n_1,n_2} & 0_{n_1,n_3} & 0_{n_1,n_4} & 0_{n_1,n_5} & 1_{n_1,n_6} & 1_{n_1,n_7}\\ 0_{n_2,n_1} & 0_{n_2,n_2} & 0_{n_2,n_3} & 0_{n_2,n_4} & 1_{n_2,n_5} & 0_{n_2,n_6} & 1_{n_2,n_7}\\ 0_{n_3,n_1} & 0_{n_3,n_2} & 0_{n_3,n_3} & 1_{n_3,n_4} & 0_{n_3,n_5} & 0_{n_3,n_6} & 1_{n_3,n_7}\\ 0_{n_4,n_1} & 0_{n_4,n_2} & 1_{n_4,n_3} & 0_{n_4,n_4} & 1_{n_4,n_5} & 1_{n_4,n_6} & 1_{n_4,n_7}\\ 0_{n_5,n_1} & 1_{n_5,n_2} & 0_{n_5,n_3} & 1_{n_5,n_4} & 0_{n_5,n_5} & 1_{n_5,n_6} & 1_{n_5,n_7}\\ 1_{n_6,n_1} & 0_{n_6,n_2} & 0_{n_6,n_3} & 1_{n_6,n_4} & 1_{n_6,n_5} & 0_{n_6,n_6} & 1_{n_6,n_7}\\ 1_{n_7,n_1} & 1_{n_7,n_2} & 1_{n_7,n_3} & 1_{n_7,n_4} & 1_{n_7,n_5} & 1_{n_7,n_6} & (1-I)_{n_7,n_7} \end{bmatrix}. \tag{13}$$

Thus, the adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)$ with row and column headings $X_1, X_2, \ldots, X_7$ is a block matrix,

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right) = \left[\begin{array}{c|c|c} 0_{n_1+n_2+n_3,n_1+n_2+n_3} & B_{n_1+n_2+n_3,n_4+n_5+n_6} & 1_{n_1+n_2+n_3,n_7} \\ \hline B^t & (1-I)_{n_4+n_5+n_6,n_4+n_5+n_6} & 1_{n_4+n_5+n_6,n_7} \\ \hline 1^t_{n_1+n_2+n_3,n_7} & 1^t_{n_4+n_5+n_6,n_7} & (1-I)_{n_7,n_7} \end{array}\right],$$

where

$$B = \begin{bmatrix} 0_{n_1,n_4} & 0_{n_1,n_5} & 1_{n_1,n_6} \\ 0_{n_2,n_4} & 1_{n_2,n_5} & 0_{n_2,n_6} \\ 1_{n_3,n_4} & 0_{n_3,n_5} & 0_{n_3,n_6} \end{bmatrix}.$$

(b) The adjacency matrix $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$ is given in (13).
By performing elementary row transformations on $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$, in the transformed matrix the number of zero rows is

$$\sum_{i=1}^{6} |X_i| - 6 = p^{k_1-1}[pq + qr + pr - (p + r + q)] - 6.$$

Thus, 0 is the eigenvalue with the multiplicity $p^{k_1-1}[pq + qr + pr - (p + r + q)] - 6$, and the nullity of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right) + I$ gives the multiplicity of an eigenvalue -1. Since the nullity of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right) + I$ is $|X_7| - 1 = p^{k_1-1} - 2$. Hence, the multiplicity of an eigenvalue -1 is $p^{k_1-1} - 2$.

(c) From (13) and Definition 2.1 the graph $\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)$ is expressed as a generalized join of two graphs,

$$\Gamma_1\left(\mathbb{Z}_{p^{k_1}qr}\right) \bigvee_{K_2} K_{n_7},$$

where $\Gamma_1\left(\mathbb{Z}_{p^{k_1}qr}\right)$ is the graph on non-nilpotent elements and K_{n_7} is the complete graph on nilpotent elements, as depicted in Figures 2 and 3.

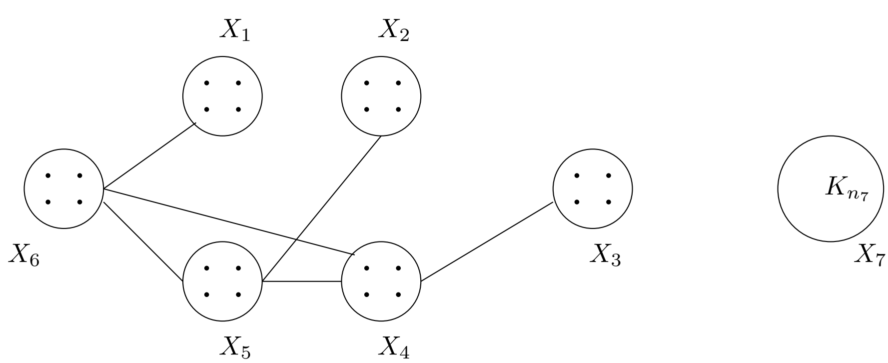

Figure 2: The induced subgraph $\Gamma_1\left(\mathbb{Z}_{p^{k_1}qr}\right)$ and complete graph K_{n_7}.

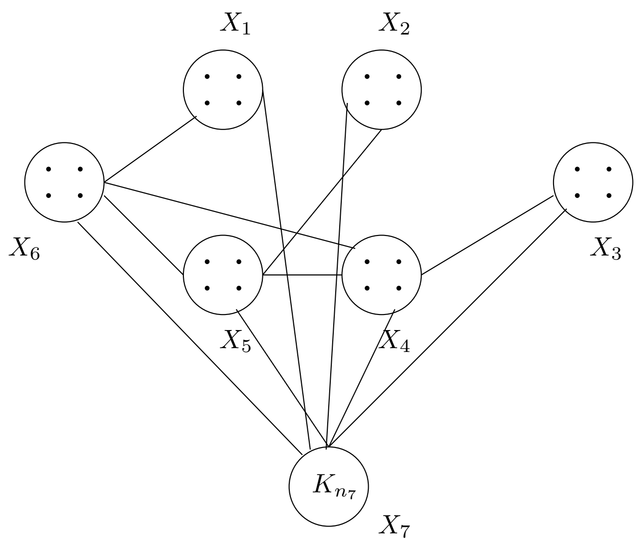

Figure 3: The generalized zero-divisor graph $\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)$.

Since the order of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$ is

$$p^{k_1-1}(pr+qr+pq-(r+q+p)+1)-1,$$

and the sum of multiplicities of the eigenvalues 0 and -1 is

$$p^{k_1-1}(pr+qr+pq-(r+q+p))-6+p^{k_1-1}-2=p^{k_1-1}(pr+qr+pq-(r+q+p)+1)-8.$$

It follows that, there are 7 remaining eigenvalues for $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}qr}\right)\right)$. Since the trace of the matrix gives the sum of eigenvalues, the sum of the remaining 7 eigenvalues is $p^{k_1-1}-2$. By Theorem 2.1, (1) and (13), we get the remaining part of the proof.
Thus, the eigenvalues of $M=[m_{ij}]_{7\times 7}$, where,

$$m_{ij}=\begin{cases} n_7-1, & i=j=7,\\ \sqrt{n_in_j}, & X_i \text{ is adjacent to } X_j,\\ 0, & \text{otherwise,}\end{cases}$$

gives the remaining 7 eigenvalues. Here, $n_i=|X_i|$.

□

Next, result determines the adjacency matrix and eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)$ where $k_1,k_2>1$.

Theorem 2.4. *Let p,r,q be primes and let $k_1>1,k_2>1$ be integers. Let the sets $X_1,X_2,\ldots,X_{11}$ be defined as in (16) and $n_i=|X_i|$ for each i;*

(a) *The adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)$ is*

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)=\left[\begin{array}{c|c|c} 0_{n_1+\ldots+n_5,n_1+\ldots+n_5} & B_{n_1+\ldots+n_5,n_6+\ldots+n_{10}} & 1_{n_1+\ldots+n_5},n_{11} \\ \hline B^t & C_{n_6+\ldots+n_{10},n_6+\ldots+n_{10}} & 1_{n_6+\ldots+n_{10}},n_{11} \\ \hline (1_{n_1+\ldots+n_5},n_{11})^t & (1_{n_6+\ldots+n_{10}},n_{11})^t & (1-I)_{n_{11},n_{11}}\end{array}\right], \tag{14}$$

where

$$B=\begin{bmatrix}0_{n_1,n_6} & 0_{n_1,n_7} & 0_{n_1,n_8} & 0_{n_1,n_9} & 1_{n_1,n_{10}}\\ 0_{n_2,n_6} & 0_{n_2,n_7} & 0_{n_2,n_8} & 1_{n_2,n_9} & 1_{n_2,n_{10}}\\ 0_{n_3,n_6} & 0_{n_3,n_7} & 1_{n_3,n_8} & 0_{n_3,n_9} & 0_{n_3,n_{10}}\\ 0_{n_4,n_6} & 1_{n_4,n_7} & 1_{n_4,n_8} & 0_{n_4,n_9} & 0_{n_4,n_{10}}\\ 1_{n_5,n_6} & 0_{n_5,n_7} & 0_{n_5,n_8} & 0_{n_5,n_9} & 0_{n_5,n_{10}}\end{bmatrix},$$

$$C=\begin{bmatrix}0_{n_6,n_6} & 1_{n_6,n_7} & 1_{n_6,n_8} & 1_{n_6,n_9} & 1_{n_6,n_{10}}\\ 1_{n_7,n_6} & 0_{n_7,n_7} & 0_{n_7,n_8} & 0_{n_7,n_9} & 1_{n_7,n_{10}}\\ 1_{n_8,n_6} & 0_{n_8,n_7} & 0_{n_8,n_8} & 1_{n_8,n_9} & 1_{n_8,n_{10}}\\ 1_{n_9,n_6} & 0_{n_9,n_7} & 1_{n_9,n_8} & 0_{n_9,n_9} & 0_{n_9,n_{10}}\\ 1_{n_{10},n_6} & 1_{n_{10},n_7} & 1_{n_{10},n_8} & 0_{n_{10},n_9} & 0_{n_{10},n_{10}}\end{bmatrix},$$

(*b*) *Zero is an eigenvalue of* $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$ *with multiplicity,*

$$p^{k_1-1}q^{k_2-1}[qr+pr+pq-(r+q+p)]-10,$$

and the multiplicity of an eigenvalue -1 *is* $p^{k_1-1}q^{k_2-1}-2$.

(*c*) *The eigenvalues of the matrix* $M=[m_{ij}]_{11\times 11}$ *where,*

$$m_{ij}=\begin{cases}n_{11}-1, & i=j=11,\\ \sqrt{n_in_j}, & X_i\sim X_j,\\ 0, & \textit{otherwise},\end{cases} \tag{15}$$

where $n_i=|X_i|$ *are the remaining* 11 *eigenvalues of* $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$.

Proof.

(a) Let $n=p^{k_1}q^{k_2}r$, where p,q,r be distinct primes and $k_1,k_2>1$ be integers. Consider a partition of $V(\Gamma'(\mathbb{Z}_n))$ into the following classes. Let,

$$\begin{aligned}
X_1&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=p^i, \quad i=1,2,\dots,k_1-1\right\},\\
X_2&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=p^{k_1}\right\},\\
X_3&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=q^i, \quad i=1,2,\dots,k_2-1\right\},\\
X_4&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=q^{k_2}\right\},\\
X_5&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=r\right\},\\
X_6&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=p^iq^j, \quad i=1,2,\dots,k_1;\quad j=1,2,\dots,k_2\right\},\\
X_7&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=p^ir, \quad i=1,2,\dots,k_1-1\right\},\\
X_8&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=p^{k_1}r\right\},\\
X_9&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=q^ir, \quad i=1,2,\dots,k_2-1\right\},\\
X_{10}&=\left\{a\in\mathbb{Z}_n\ :\ (a,n)=q^{k_2}r\right\},\\
X_{11}&=\left\{a\in\mathbb{Z}_n\ :\ a=kprq, \quad k=1,2,\dots,p^{k_1-1}q^{k_2-1}-1\right\}.
\end{aligned} \tag{16}$$

Observe that, X_{11} contains the nilpotent elements in $\mathbb{Z}_{p^{k_1}q^{k_2}r}$ and all the sets $X_1, X_2, \ldots, X_{11}$ are mutually disjoint. Hence,

$$\{X_1 \cup X_2 \cup \ldots \cup X_{11}\}, \tag{17}$$

form a partition of $V\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$. Next, we find the cardinality of the sets $X_1, X_2, \ldots, X_{11}$. Suppose that, $x \in X_1$. Then, $gcd(x, n) = p^i$. Therefore,

$$\begin{aligned} n_1 = |X_1| &= \phi\left(p^{k_1-1}q^{k_2}r\right) + \cdots + \phi\left(pq^{k_2}r\right) \\ &= p^{k_1-2}q^{k_2-1}(p-1)(r-1)(q-1) + \cdots + q^{k_2-1}(p-1)(r-1)(q-1) \\ &= \left(p^{k_1-1}-1\right)q^{k_2-1}(r-1)(q-1). \end{aligned}$$

Similarly,

$$\begin{aligned} n_2 &= |X_2| = q^{k_2-1}(r-1)(q-1), & n_3 &= |X_3| = (r-1)(q-1)p^{k_1-1}(p-1), \\ n_4 &= |X_4| = p^{k_1-1}(r-1)(p-1), & n_5 &= |X_5| = q(q-1)p^{k_1-1}(p-1), \\ n_6 &= |X_6| = p^{k_1-1}q^{k_2-1}(r-1), & n_7 &= |X_7| = (p-1)q^{k_2-1}(q-1), \\ n_8 &= |X_8| = q^{k_2-1}(q-1), & n_9 &= |X_9| = (q-1)p^{k_1-1}(p-1), \\ n_{10} &= |X_{10}| = p^{k_1-1}(p-1), & n_{11} &= |X_{11}| = p^{k_1-1}q^{k_2-1} - 1, \end{aligned}$$

$$\phi(n) = \phi\left(p^{k_1}q^{k_2}r\right) = p^{k_1-1}(p-1)(r-1)q^{k_2-1}(q-1),$$

and

$$\begin{aligned} \left|V\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)\right| = n - \phi(n) - 1 &= p^{k_1}q^{k_2}r - p^{k_1-1}(p-1)q^{k_2-1}(q-1)(r-1) - 1 \\ &= p^{k_1-1}q^{k_2-1}(pr + qr + pq - (r+q+p) + 1) - 1. \end{aligned}$$

We get the following relationships among the vertices,

$$\begin{array}{llllll} X_1 \sim X_{10}, & X_1 \sim X_{11}, & & & & \\ X_2 \sim X_9, & X_2 \sim X_{10}, & X_2 \sim X_{11}, & & & \\ X_3 \sim X_8, & X_3 \sim X_{11}, & & & & \\ X_4 \sim X_7, & X_4 \sim X_8, & X_4 \sim X_{11}, & & & \\ X_5 \sim X_6, & X_5 \sim X_{11}, & & & & \\ X_6 \sim X_5, & X_6 \sim X_7, & X_6 \sim X_8, & X_6 \sim X_9, & X_6 \sim X_{10}, & X_6 \sim X_{11}, \\ X_7 \sim X_4, & X_7 \sim X_6, & X_7 \sim X_{10}, & X_7 \sim X_{11}, & & \\ X_8 \sim X_3, & X_8 \sim X_4, & X_8 \sim X_6, & X_8 \sim X_9, & X_8 \sim X_{10}, & X_8 \sim X_{11}, \\ X_9 \sim X_2, & X_9 \sim X_6, & X_9 \sim X_8, & X_9 \sim X_{11}, & & \\ X_{10} \sim X_1, & X_{10} \sim X_2, & X_{10} \sim X_6, & X_{10} \sim X_7, & X_{10} \sim X_8, & X_{10} \sim X_{11}, \\ X_{11} \sim X_1, & X_{11} \sim X_2, \ldots, & X_{11} \sim X_{11}. & & & \end{array}$$

Hence, the adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)$ with row and column headings $X_1, X_2, \ldots, X_{11}$ is

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right) = \left[\begin{array}{c|c|c} 0_{n_1+\ldots+n_5, n_1+\ldots+n_5} & B_{n_1+\ldots+n_5, n_6+\ldots+n_{10}} & 1_{n_1+\ldots+n_5}, n_{11} \\ \hline B^t & C_{n_6+\ldots+n_{10}, n_6+\ldots+n_{10}} & 1_{n_6+\ldots+n_{10}}, n_{11} \\ \hline (1_{n_1+\ldots+n_5}, n_{11})^t & (1_{n_6+\ldots+n_{10}}, n_{11})^t & (1-I)_{n_{11}, n_{11}} \end{array}\right],$$

where

$$B = \begin{bmatrix} 0_{n_1,n_6} & 0_{n_1,n_7} & 0_{n_1,n_8} & 0_{n_1,n_9} & 1_{n_1,n_{10}} \\ 0_{n_2,n_6} & 0_{n_2,n_7} & 0_{n_2,n_8} & 1_{n_2,n_9} & 1_{n_2,n_{10}} \\ 0_{n_3,n_6} & 0_{n_3,n_7} & 1_{n_3,n_8} & 0_{n_3,n_9} & 0_{n_3,n_{10}} \\ 0_{n_4,n_6} & 1_{n_4,n_7} & 1_{n_4,n_8} & 0_{n_4,n_9} & 0_{n_4,n_{10}} \\ 1_{n_5,n_6} & 0_{n_5,n_7} & 0_{n_5,n_8} & 0_{n_5,n_9} & 0_{n_5,n_{10}} \end{bmatrix},$$

$$C = \begin{bmatrix} 0_{n_6,n_6} & 1_{n_6,n_7} & 1_{n_6,n_8} & 1_{n_6,n_9} & 1_{n_6,n_{10}} \\ 1_{n_7,n_6} & 0_{n_7,n_7} & 0_{n_7,n_8} & 0_{n_7,n_9} & 1_{n_7,n_{10}} \\ 1_{n_8,n_6} & 0_{n_8,n_7} & 0_{n_8,n_8} & 1_{n_8,n_9} & 1_{n_8,n_{10}} \\ 1_{n_9,n_6} & 0_{n_9,n_7} & 1_{n_9,n_8} & 0_{n_9,n_9} & 0_{n_9,n_{10}} \\ 1_{n_{10},n_6} & 1_{n_{10},n_7} & 1_{n_{10},n_8} & 0_{n_{10},n_9} & 0_{n_{10},n_{10}} \end{bmatrix}.$$

(b) The adjacency matrix $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$ is given in (14). Similar to the proof of Theorem 18 (b), by performing elementary row transformations on $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$, in the transformed matrix the number of zero rows is

$$\sum_{i=1}^{10} |X_i| - 10 = p^{k_1-1}q^{k_2-1}\left[pq + qr + pr - (p + r + q)\right] - 10.$$

Hence, the multiplicity of an eigenvalue 0 is

$$p^{k_1-1}q^{k_2-1}\left[pq + qr + pr - (p + r + q)\right] - 10.$$

The nullity of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right) + I$ is the multiplicity of an eigenvalue -1. Hence, the multiplicity of an eigenvalue -1 is $|X_{11}| - 1 = p^{k_1-1}q^{k_2-1} - 2$.

(c) Since the order of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$ is

$$p^{k_1-1}q^{k_2-1}(pr + qr + pq - (r + q + p) + 1) - 1,$$

and the sum of multiplicities of eigenvalues 0 and -1 is

$$\begin{aligned} p^{k_1-1}q^{k_2-1}(pr + qr + pq - (r + q + p)) - 10 + p^{k_1-1}q^{k_2-1} - 2 \\ = p^{k_1-1}q^{k_2-1}(pr + qr + pq - (r + q + p) + 1) - 12. \end{aligned}$$

Hence, there are 11 more eigenvalues of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right)$. Since the trace of the matrix gives the sum of the eigenvalues of a matrix. The sum of remaining 11 eigenvalues is $p^{k_1-1}q^{k_2-1}-2$. By Theorem 2.1, (1) and (14), we get the remaining part of the proof.

Hence, the eigenvalues of the matrix $M = [m_{ij}]_{11\times 11}$ gives the remaining 11 eigenvalues, where,

$$m_{ij} = \begin{cases} n_{11} - 1, & i = j = 11, \\ \sqrt{n_i n_j}, & X_i \sim X_j, \\ 0, & \text{otherwise}, \end{cases}$$

and $n_i = |X_i|$.

□

Now, we obtain the eigenvalues of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

Theorem 2.5. *Let p, r, q be primes and $k_1, k_2, k_3 > 1$ be integers. Let the sets $Y_1, Y_2, \ldots, Y_{19}$ be given in (20) and $n_i = |Y_i|$;*

(a) *The adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ is*

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = \left[\begin{array}{c|c|c|c|c} 0_{n_1+\ldots+n_6, n_1+\ldots+n_6} & B_1 & B_2 & B_3 & 1_{n_1+\ldots+n_6}, n_{19} \\ \hline B_1^t & 0 & C_1 & C_2 & 1_{n_7+\ldots+n_{10}}, n_{19} \\ \hline B_2^t & C_1^t & 0 & C_3 & 1_{n_{11}+\ldots+n_{14}}, n_{19} \\ \hline B_3^t & C_2^t & C_3^t & 0 & 1_{n_{15}+\ldots+n_{18}}, n_{19} \\ \hline 1^t & 1^t & 1^t & 1^t & (1-I)_{n_{19},n_{19}} \end{array}\right], \tag{18}$$

where

$$B_1 = \begin{bmatrix} 0_{n_1,n_7} & 0_{n_1,n_8} & 0_{n_1,n_9} & 0_{n_1,n_{10}} \\ 0_{n_2,n_7} & 0_{n_2,n_8} & 0_{n_2,n_9} & 0_{n_2,n_{10}} \\ 0_{n_3,n_7} & 0_{n_3,n_8} & 0_{n_3,n_9} & 0_{n_3,n_{10}} \\ 0_{n_4,n_7} & 0_{n_4,n_8} & 0_{n_4,n_9} & 0_{n_4,n_{10}} \\ 0_{n_5,n_7} & 0_{n_5,n_8} & 0_{n_5,n_9} & 1_{n_5,n_{10}} \\ 1_{n_6,n_7} & 1_{n_6,n_8} & 1_{n_6,n_9} & 1_{n_6,n_{10}} \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0_{n_7,n_{11}} & 0_{n_7,n_{12}} & 1_{n_7,n_{13}} & 1_{n_7,n_{14}} \\ 0_{n_8,n_{11}} & 0_{n_8,n_{12}} & 1_{n_8,n_{13}} & 1_{n_8,n_{14}} \\ 1_{n_9,n_{11}} & 1_{n_9,n_{12}} & 1_{n_9,n_{13}} & 1_{n_9,n_{14}} \\ 1_{n_{10},n_{11}} & 1_{n_{10},n_{12}} & 1_{n_{10},n_{13}} & 1_{n_{10},n_{14}} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0_{n_1,n_{11}} & 0_{n_1,n_{12}} & 0_{n_1,n_{13}} & 0_{n_1,n_{14}} \\ 0_{n_2,n_{11}} & 0_{n_2,n_{12}} & 0_{n_2,n_{13}} & 0_{n_2,n_{14}} \\ 0_{n_3,n_{11}} & 0_{n_3,n_{12}} & 0_{n_3,n_{13}} & 1_{n_3,n_{14}} \\ 1_{n_4,n_{11}} & 1_{n_4,n_{12}} & 1_{n_4,n_{13}} & 1_{n_4,n_{14}} \\ 0_{n_5,n_{11}} & 0_{n_5,n_{12}} & 0_{n_5,n_{13}} & 0_{n_5,n_{14}} \\ 0_{n_6,n_{11}} & 0_{n_6,n_{12}} & 0_{n_6,n_{13}} & 0_{n_6,n_{14}} \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0_{n_7,n_{15}} & 0_{n_7,n_{16}} & 1_{n_7,n_{17}} & 0_{n_7,n_{18}} \\ 1_{n_8,n_{15}} & 1_{n_8,n_{16}} & 1_{n_8,n_{17}} & 0_{n_8,n_{18}} \\ 0_{n_9,n_{15}} & 0_{n_9,n_{16}} & 1_{n_9,n_{17}} & 1_{n_9,n_{18}} \\ 1_{n_{10},n_{15}} & 1_{n_{10},n_{16}} & 1_{n_{10},n_{17}} & 1_{n_{10},n_{18}} \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0_{n_1,n_{15}} & 0_{n_1,n_{16}} & 0_{n_1,n_{17}} & 1_{n_1,n_{18}} \\ 1_{n_2,n_{15}} & 1_{n_2,n_{16}} & 1_{n_2,n_{17}} & 1_{n_2,n_{18}} \\ 0_{n_3,n_{15}} & 0_{n_3,n_{16}} & 0_{n_3,n_{17}} & 0_{n_3,n_{18}} \\ 0_{n_4,n_{15}} & 0_{n_4,n_{16}} & 0_{n_4,n_{17}} & 0_{n_4,n_{18}} \\ 0_{n_5,n_{15}} & 0_{n_5,n_{16}} & 0_{n_5,n_{17}} & 0_{n_5,n_{18}} \\ 0_{n_6,n_{15}} & 0_{n_6,n_{16}} & 0_{n_6,n_{17}} & 0_{n_6,n_{18}} \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0_{n_{11},n_{15}} & 1_{n_{11},n_{16}} & 0_{n_{11},n_{17}} & 1_{n_{11},n_{18}} \\ 1_{n_{12},n_{15}} & 1_{n_{12},n_{16}} & 1_{n_{12},n_{17}} & 1_{n_{12},n_{18}} \\ 1_{n_{13},n_{15}} & 1_{n_{13},n_{16}} & 1_{n_{13},n_{17}} & 1_{n_{13},n_{18}} \\ 1_{n_{14},n_{15}} & 1_{n_{14},n_{16}} & 1_{n_{14},n_{17}} & 1_{n_{14},n_{18}} \end{bmatrix}.$$

(b) *Zero is an eigenvalue of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$ with multiplicity,*

$$p^{k_1-1}q^{k_2-1}r^{k_3-1}[pq + pr + qr - (p + r + q)] - 18,$$

and -1 is an eigenvalue with multiplicity $p^{k_1-1}q^{k_2-1}r^{k_3-1} - 2$.

(c) *The eigenvalues of the matrix $M = [m_{ij}]_{19\times 19}$ are the remaining 19 eigenvalues of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$, where,*

$$m_{ij} = \begin{cases} n_{19} - 1, & i = j = 19, \\ \sqrt{n_i n_j}, & Y_i \sim Y_j, \\ 0, & \textit{otherwise}, \end{cases} \tag{19}$$

and $n_i = |Y_i|$.

Proof.

(a) Let $m = p^{k_1}q^{k_2}r^{k_3}$ and p, q, r be distinct primes. Partition $V(\Gamma'(\mathbb{Z}_n))$ into the following classes. Let,

$$\begin{aligned}
Y_1 &= \{y \in \mathbb{Z}_m : (y,m) = p^i, \quad i = 1,2,\ldots,k_1-1\},\\
Y_2 &= \{y \in \mathbb{Z}_m : (y,m) = p^{k_1}\},\\
Y_3 &= \{y \in \mathbb{Z}_m : (y,m) = q^i, \quad i = 1,2,\ldots,k_2-1\},\\
Y_4 &= \{y \in \mathbb{Z}_m : (y,m) = q^{k_2}\},\\
Y_5 &= \{y \in \mathbb{Z}_m : (y,m) = r^i, \quad i = 1,2,\ldots,k_3-1\},\\
Y_6 &= \{y \in \mathbb{Z}_m : (y,m) = r^{k_3}\},\\
Y_7 &= \{y \in \mathbb{Z}_m : (y,m) = p^iq^j, \quad i = 1,2,\ldots,k_1-1; \quad j = 1,2,\ldots,k_2-1\},\\
Y_8 &= \{y \in \mathbb{Z}_m : (y,m) = p^{k_1}q^j, \quad j = 1,2,\ldots,k_2-1\},\\
Y_9 &= \{y \in \mathbb{Z}_m : (y,m) = p^iq^{k_2}, \quad i = 1,2,\ldots,k_1-1\},\\
Y_{10} &= \{y \in \mathbb{Z}_m : (y,m) = p^{k_1}q^{k_2}\},\\
Y_{11} &= \{y \in \mathbb{Z}_m : (y,m) = p^ir^j, \quad i = 1,2,\ldots,k_1-1; \quad j = 1,2,\ldots,k_3-1\},\\
Y_{12} &= \{y \in \mathbb{Z}_m : (y,m) = p^{k_1}r^j, \quad j = 1,2,\ldots,k_3-1\},\\
Y_{13} &= \{y \in \mathbb{Z}_m : (y,m) = p^ir^{k_3}, \quad i = 1,2,\ldots,k_1-1\},\\
Y_{14} &= \{y \in \mathbb{Z}_m : (y,m) = p^{k_1}r^{k_3}\},\\
Y_{15} &= \{y \in \mathbb{Z}_m : (y,m) = q^ir^j, \quad i = 1,2,\ldots,k_2-1; \quad j = 1,2,\ldots,k_3-1\},\\
Y_{16} &= \{y \in \mathbb{Z}_m : (y,m) = q^{k_2}r^j, \quad j = 1,2,\ldots,k_3-1\},\\
Y_{17} &= \{y \in \mathbb{Z}_m : (y,m) = q^ir^{k_3}, \quad i = 1,2,\ldots,k_2-1\},\\
Y_{18} &= \{y \in \mathbb{Z}_m : (y,m) = q^{k_2}r^{k_3}\},\\
Y_{19} &= \{y \in \mathbb{Z}_m : y = kprq, \quad k = 1,2,\ldots,p^{k_1-1}q^{k_2-1}r^{k_3-1}-1\}.
\end{aligned} \tag{20}$$

Observe that, Y_{19} consist of nilpotent elements in $\mathbb{Z}_m$ and all the sets $Y_1, Y_2, \ldots, Y_{19}$ are mutually disjoint. Hence,

$$\{Y_1 \cup Y_2 \cup \ldots \cup Y_{19}\}, \tag{21}$$

form a partition of $V(\Gamma'(\mathbb{Z}_m))$. Next, we find the cardinality of the sets $Y_1, Y_2, \ldots, Y_{19}$. Suppose that, $y \in Y_1$. Then, $(y,m) = p$. Therefore,

$$\begin{aligned}
n_1 = |Y_1| &= \phi(p^{k_1-1}q^{k_2}r^{k_3}) + \cdots + \phi(pq^{k_2}r^{k_3})\\
&= p^{k_1-2}(p-1)(q-1)q^{k_2-1}r^{k_3-1}(r-1) + \cdots + (p-1)q^{k_2-1}r^{k_3-1}(r-1)(q-1)\\
&= (p^{k_1-1}-1)(q-1)q^{k_2-1}r^{k_3-1}(r-1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
n_2 &= |Y_2| = q^{k_2-1}(q-1)r^{k_3-1}(r-1),\\
n_3 &= |Y_3| = (q^{k_2-1}-1)p^{k_1-1}r^{k_3-1}(r-1)(p-1),\\
n_4 &= |Y_4| = (p-1)p^{k_1-1}r^{k_3-1}(r-1),\\
n_5 &= |Y_5| = (r^{k_3-1}-1)(p-1)p^{k_1-1}q^{k_2-1}(q-1),\\
n_6 &= |Y_6| = (p-1)p^{k_1-1}q^{k_2-1}(q-1),
\end{aligned}$$

$$
\begin{aligned}
n_7 &= |Y_7| = (p-1)r^{k_3-1}(r-1)(q-1),\\
n_8 &= |Y_8| = r^{k_3-1}(r-1)(q-1),\\
n_9 &= |Y_9| = r^{k_3-1}(r-1)(p-1),\\
n_{10} &= |Y_{10}| = r^{k_3-1}(r-1),\\
n_{11} &= |Y_{11}| = (p-1)(r-1)q^{k_2-1}(q-1),\\
n_{12} &= |Y_{12}| = (r-1)q^{k_2-1}(q-1),\\
n_{13} &= |Y_{13}| = (q-1)q^{k_2-1}(p-1),\\
n_{14} &= |Y_{14}| = q^{k_2-1}(q-1),\\
n_{15} &= |Y_{15}| = (p-1)p^{k_1-1}(r-1)(q-1),\\
n_{16} &= |Y_{16}| = (r-1)p^{k_1-1}(p-1),\\
n_{17} &= |Y_{17}| = (q-1)p^{k_1-1}(p-1),\\
n_{18} &= |Y_{18}| = (p-1)p^{k_1-1},\\
n_{19} &= |Y_{19}| = p^{k_1-1}q^{k_2-1}r^{k_3-1} - 1.
\end{aligned}
$$

$$\phi(m) = \phi(p^{k_1}q^{k_2}r^{k_3}) = p^{k_1-1}(p-1)q^{k_2-1}(q-1)r^{k_3-1}(r-1), \quad \text{and}$$
$$|Z^*(\mathbb{Z}_m)| = m - \phi(m) - 1 = p^{k_1-1}q^{k_2-1}r^{k_3-1}(pr+qr+pq-(r+q+p)+1) - 1,$$

which is the order of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$. Observe the following adjacency relations;

$Y_1 \sim Y_{18}, Y_{19};$
$Y_2 \sim Y_{15}, Y_{16}, \ldots, Y_{19};$
$Y_3 \sim Y_{14}, Y_{19};$
$Y_4 \sim Y_{11}, \ldots, Y_{14}, Y_{19};$
$Y_5 \sim Y_{10}, Y_{19};$
$Y_6 \sim Y_7, \ldots, Y_{10}, Y_{19};$
$Y_7 \sim Y_6, Y_{13}, Y_{14}, Y_{17}, Y_{19};$
$Y_8 \sim Y_6, Y_{13}, \ldots, Y_{17}, Y_{19};$
$Y_9 \sim Y_6, Y_{11}, \ldots, Y_{14}, Y_{17}, \ldots, Y_{19};$
$Y_{10} \sim Y_5, Y_6, Y_{11}, \ldots, Y_{19};$
$Y_{11} \sim Y_4, Y_9, Y_{10}, Y_{16}, Y_{18}, Y_{19};$
$Y_{12} \sim Y_4, Y_9, Y_{10}, Y_{15}, \ldots, Y_{19};$
$Y_{13} \sim Y_4, Y_9, Y_{10}, Y_{15}, \ldots, Y_{19};$
$Y_{14} \sim Y_3, Y_4, Y_7, \ldots, Y_{10}, Y_{15}, \ldots, Y_{19};$
$Y_{15} \sim Y_2, Y_8, Y_{10}, Y_{12}, Y_{14}, Y_{19};$
$Y_{16} \sim Y_2, Y_8, Y_{10}, \ldots, Y_{14}, Y_{19};$
$Y_{17} \sim Y_2, Y_7, \ldots, Y_{10}, Y_{12}, Y_{19};$
$Y_{18} \sim Y_1, Y_2, Y_7, \ldots, Y_{10}, \ldots, Y_{14}, Y_{19};$
$Y_{19} \sim Y_1, Y_2, \ldots, Y_{19}.$

The adjacency matrix of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ with row and column headings $Y_1, Y_2, \ldots, Y_{19}$ is

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = [a_{ij}],$$

where

$$a_{ij} = \begin{cases} (1-I)_{n_i \times n_j}, & i = j = 19,\\ 1_{n_i \times n_j}, & Y_i \sim Y_j,\\ 0_{n_i \times n_j}, & \text{otherwise.} \end{cases}$$

Hence,

$$A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = \left[\begin{array}{c|c|c|c|c} 0_{n_1+\ldots+n_6, n_1+\ldots+n_6} & B_1 & B_2 & B_3 & 1_{n_1+\ldots+n_6, n_{19}}\\ \hline B_1^t & 0 & C_1 & C_2 & 1_{n_7+\ldots+n_{10}, n_{19}}\\ \hline B_2^t & C_1^t & 0 & C_3 & 1_{n_{11}+\ldots+n_{14}, n_{19}}\\ \hline B_3^t & C_2^t & C_3^t & 0 & 1_{n_{15}+\ldots+n_{18}, n_{19}}\\ \hline 1^t & 1^t & 1^t & 1^t & (1-I)_{n_{19}, n_{19}} \end{array}\right],$$

where $B_1, B_2, B_3, C_1, C_2, C_3$ are matrices as in (18).

(b) The adjacency matrix $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$ is given in (18). Similar to the proof of Theorem 18 (b), by performing elementary row transformations on $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$, the number of zero rows in the transformed matrix is

$$\sum_{i=1}^{18}|Y_i| - 18 = p^{k_1-1}r^{k_3-1}q^{k_2-1}[pr+qr+pq-(r+q+p)] - 18.$$

Hence, the multiplicity of an eigenvalue 0 is

$$p^{k_1-1}r^{k_3-1}q^{k_2-1}[pr+qr+pq-(r+q+p)] - 18.$$

The nullity of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)+I$ gives the multiplicity of an eigenvalue -1. Hence, the multiplicity of an eigenvalue -1 is $|Y_{19}|-1 = p^{k_1-1}q^{k_2-1}r^{k_3-1}-2$.

(c) Since the order of matrix $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$ is $p^{k_1-1}q^{k_2-1}r^{k_3-1}(pr+qr+pq-(r+q+p)+1)-1$ and the sum of multiplicities of eigenvalues 0 and -1 is

$$\begin{aligned} &p^{k_1-1}q^{k_2-1}r^{k_3-1}(pr+qr+pq-(r+q+p)) - 18 + p^{k_1-1}q^{k_2-1}r^{k_3-1} - 2 \\ &= p^{k_1-1}r^{k_3-1}q^{k_2-1}(pr+qr+pq-(r+q+p)+1) - 20. \end{aligned}$$

Hence, there are 19 more eigenvalues of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$. Since the sum of eigenvalues is the trace of the matrix. Hence, $p^{k_1-1}q^{k_2-1}r^{k_3-1}-2$ gives the sum of remaining 19 eigenvalues. We get the remaining part of the proof by Theorem 2.1, (1) and (18).

Thus, the eigenvalues of the matrix $M=[m_{ij}]_{19\times 19}$ are the remaining 19 eigenvalues where,

$$m_{ij} = \begin{cases} n_{19}-1, & i=j=19, \\ \sqrt{n_i n_j}, & Y_i \sim Y_j, \\ 0, & \text{otherwise}, \end{cases}$$

and $n_i = |Y_i|$.

□

3 Properties of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$

This section determines the girth, clique number, diameter and stability number of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$. Furthermore, we prove Beck's conjecture for $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

Theorem 3.1. *The clique number of $\Gamma'(\mathbb{Z}_{prq})$ is $\omega(\Gamma'(\mathbb{Z}_{prq}))=3$ and the stability number is*

$$\alpha(\Gamma'(\mathbb{Z}_{prq})) = pr+pq+qr-2(r+p+q)+3.$$

Proof. Let X_i be as defined in Theorem (18) and (4). From the Figure 1 observe that, if $x\in X_4$, $y\in X_5$, $z\in X_6$, then, $\{x,y,z\}$ form a clique that have the maximum size. Hence, $\omega(\Gamma'(\mathbb{Z}_{prq}))=3$.

For any $x, y \in X_1 \cup X_2 \cup X_3$, $x^m y \neq 0$ and $y^m x \neq 0$. Therefore, there is no edge between any two vertices of $X_1 \cup X_2 \cup X_3$ in $\Gamma'(\mathbb{Z}_{prq})$. Hence, the stability number of $\Gamma'(\mathbb{Z}_{prq})$ is

$$\begin{aligned}\alpha(\Gamma'(\mathbb{Z}_{prq})) &= |X_1| + |X_2| + |X_3| \\ &= (r-1)(q-1) + (p-1)(r-1) + (q-1)(p-1) \\ &= pr + qr + pq - 2(r+q+p) + 3.\end{aligned}$$

□

Corollary 3.1. *The girth of $\Gamma'(\mathbb{Z}_{prq})$ is $gr(\Gamma'(\mathbb{Z}_{prq})) = 3$ and diameter of $\Gamma'(\mathbb{Z}_{prq})$ is*

$$diam(\Gamma'(\mathbb{Z}_{prq})) = 3.$$

Theorem 3.2. *The clique number of $\Gamma'(\mathbb{Z}_{p^{k_1}qr})$ is $\omega(\Gamma'(\mathbb{Z}_{p^{k_1}qr})) = p^{k_1-1} + 2$ and stability number is*

$$\alpha(\Gamma'(\mathbb{Z}_{p^{k_1}qr})) = p^{k_1-1}[pr + pq + qr - 2(r+p+q) + 3], \quad \textit{for } k_1 > 1.$$

Proof. Let X_i be as defined in Theorem 2.3 and (4). Since X_7 contains the nilpotent elements, the subgraph formed by the vertices in X_7 correspond to the complete subgraph given by the principal submatrix $[1-I]$ of $A(\Gamma'(\mathbb{Z}_{p^{k_1}qr}))$.

Let $x \in X_4$, $y \in X_5$, $z \in X_6$. Then, the subgraph induced by the vertices in $X_7 \cup \{x, y, z\}$ forms a complete subgraph of maximum order. Hence, the clique number of $\Gamma'(\mathbb{Z}_{p^{k_1}qr})$ is

$$\omega(\Gamma'(\mathbb{Z}_{p^{k_1}qr})) = |X_7| + 3 = p^{k_1-1} - 1 + 3 = p^{k_1-1} + 2.$$

For any $x, y \in X_1 \cup X_2 \cup X_3$, we have $x^m y \neq 0$ and $y^m x \neq 0$ for any positive integer m. Therefore, there is no edge between any two vertices of $X_1 \cup X_2 \cup X_3$ in $\Gamma'(\mathbb{Z}_{p^{k_1}qr})$. Hence, the stability number of $\Gamma'(\mathbb{Z}_{p^{k_1}qr})$ is

$$\begin{aligned}\alpha(\Gamma'(\mathbb{Z}_{p^{k_1}qr})) &= |X_1| + |X_2| + |X_3| \\ &= (q-1)p^{k_1-1}(r-1) + (r-1)p^{k_1-1}(p-1) + (p-1)p^{k_1-1}(q-1) \\ &= p^{k_1-1}[pr + qr + pq - 2(r+q+p) + 3].\end{aligned}$$

□

Corollary 3.2. *The girth and diameter of $\Gamma'(\mathbb{Z}_{p^{k_1}qr})$ are*

$$gr(\Gamma'(\mathbb{Z}_{p^{k_1}qr})) = 3, \quad \textit{and} \quad diam(\Gamma'(\mathbb{Z}_{p^{k_1}qr})) = 3.$$

Theorem 3.3. *The clique number of $\Gamma'(\mathbb{Z}_{p^{k_1}q^{k_2}r})$ is*

$$\omega(\Gamma'(\mathbb{Z}_{p^{k_1}q^{k_2}r})) = p^{k_1-1}q^{k_2-1} + 2,$$

and stability number is

$$\alpha(\Gamma'(\mathbb{Z}_{p^{k_1}q^{k_2}r})) = p^{k_1-1}q^{k_2-1}[pr + qr + pq - 2(r+q+p) + 3],$$

where $k_1, k_2 > 1$.

Proof. Let X_i be as defined in Theorem 2.4, (16). Since X_{11} consists of nilpotent elements, the induced subgraph by the vertices in X_{11} corresponds to the complete subgraph given by the principal submatrix $[1-I]$ of $A(\Gamma'(\mathbb{Z}_{p^{k_1}q^{k_2}r}))$.

Let $x \in X_6$, $y \in X_8$, $z \in X_{10}$. Then, the subgraph induced by the vertices in $X_{11} \cup \{x, y, z\}$ gives a complete subgraph of maximum order. Hence, the clique number,

$$\omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right) = |X_{11}| + 3 = p^{k_1-1}q^{k_2-1} - 1 + 3 = p^{k_1-1}q^{k_2-1} + 2.$$

For any $x, y \in X_1 \cup X_2 \cup \ldots \cup X_5$, we have $x^m y \neq 0$ and $y^m x \neq 0$ for any positive integer m. Therefore, there is no edge between any two vertices of $X_1 \cup X_2 \cup \ldots \cup X_5$ in $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)$. Hence, the stability number of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)$ is

$$\begin{aligned}
\alpha\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r}\right)\right) &= |X_1| + |X_2| + |X_3| + |X_4| + |X_5| \\
&= (p-1)(r-1)q^{k_2-1}(q-1) + (r-1)q^{k_2-1}(q-1) + (p-1)p^{k_1-1}(r-1)(q-1) \\
&\quad + (r-1)p^{k_1-1}(p-1) + p^{k_1-1}q^{k_1-1}(q-1)(p-1) \\
&= p^{k_1-1}q^{k_2-1}[pr + qr + pq - 2(r + q + p) + 3].
\end{aligned}$$

□

Theorem 3.4. *Let $k_1, k_2, k_3 > 1$ be integers. Then, the clique number of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ is*

$$\omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = p^{k_1-1}q^{k_2-1}r^{k_3-1} + 2,$$

and stability number is

$$\alpha\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = p^{k_1-1}q^{k_2-1}r^{k_3-1}[pr + qr + pq - 2(r + q + p) + 3].$$

Proof. Let Y_i be as defined in Theorem 2.5, (20). Since Y_{19} consists of nilpotent elements. The subgraph corresponding to the vertices in Y_{19} gives the complete graph represented by the principal submatrix $\left[1 - I\right]$ of $A\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$.

Let $x \in Y_{10}$, $y \in Y_{14}$, $z \in Y_{18}$. Then, the subgraph induced by the vertices in $Y_{19} \cup \{x, y, z\}$ forms a complete subgraph of maximum order. Thus,

$$\omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = |Y_{19}| + 3 = p^{k_1-1}q^{k_2-1}r^{k_3-1} - 1 + 3 = p^{k_1-1}q^{k_2-1}r^{k_3-1} + 2.$$

Furthermore, for any $x, y \in Y_1 \cup Y_2 \cup \ldots \cup Y_6$, we have $x^m y \neq 0$ and $y^m x \neq 0$ for any positive integer m. Therefore, there is no edge between any two vertices of $Y_1 \cup Y_2 \cup \ldots \cup Y_6$ in $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$. This means that the stability number $\alpha\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right)$ can be found by summing the sizes of the independent sets $Y_1, Y_2, \ldots, Y_6$ as these sets have no edges between them. Thus, the stability number of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ is

$$\begin{aligned}
\alpha\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) &= \sum_{i=1}^{6} |Y_i| \\
&= p^{k_1-1}q^{k_2-1}r^{k_3-1}[pr + qr + pq - 2(r + q + p) + 3].
\end{aligned}$$

□

Corollary 3.3. *Let $k_1, k_2, k_3 > 1$ be integers. The girth and diameter of $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ are*

$$gr\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = 3 \quad \text{and} \quad diam\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = 2.$$

Proof. From the earlier result, we know that,

$$\omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = p^{k_1-1}q^{k_2-1}r^{k_3-1} + 2 > 3.$$

Specifically, if we consider $x \in Y_{14}$, $y \in Y_{18}$, $z \in Y_{19}$, then, the set $\{x, y, z\}$ gives a clique in $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$. Hence, $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$ contains a triangle. Thus, $gr\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = 3$. If z, w are two distinct nilpotent elements, then, $d(z, w) = 1$. If z is a nilpotent element and w is not a nilpotent element, then, $d(z, w) = 1$. If both z, w are non-nilpotent elements and they are adjacent, then, $d(z, w) = 1$. If both z, w are non-nilpotent elements and they are not adjacent, then, $z \leftrightarrow u \leftrightarrow w$ is a path, for any nilpotent element u. Therefore, $diam\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = 2$. □

We conclude this section by proving Beck's conjecture for $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

Theorem 3.5. *Let k_1, k_2, k_3 be positive integers p, q, r be primes. Then,*

$$\chi\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = \omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right).$$

Proof. Consider the case $k_1 > 1$, $k_2 > 1$, $k_3 > 1$. Let Y_i, $i = 1, 2, \ldots, 19$ be the sets as defined in Theorem 2.5 (20). By Theorem 3.4, we have

$$\omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = p^{k_1-1}q^{k_2-1}r^{k_3-1} + 2.$$

Now, observe that the vertices in $Y_{19} \cup \{x, y, z\}$, where $x \in Y_{10}$, $y \in Y_{14}$, $z \in Y_{18}$ form a clique of the maximum size in $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$. Hence, we use different colors to color the vertices in this set. For any Y_i, $i \neq 19$, no element in Y_i is adjacent to any other element in Y_i. Hence, we use the same color to color the vertices in Y_i. Also, as Y_{19} contains nilpotent elements we will not use any color used in the vertices of Y_{19} to the remaining vertices. Let c_1, c_2, c_3 be the colors used to color the vertices in Y_{10}, Y_{14}, Y_{18} respectively. We color the remaining vertices with these three colors as follows: Consider any $w \in Y_i$, $i \neq 19, 10, 14, 18$ then, $Y_i \nsim Y_j$ for some $j = 10, 14, 18$ otherwise $Y_{19} \cup \{x, y, z, w\}$ form a clique, a contradiction to Theorem 3.4. Hence, w is not adjacent to at least one of x, y, z. Suppose that, w is not adjacent to x. Then, we color vertex w with the same color as that of x. Therefore,

$$\chi\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = |Y_{19}| + 3.$$

Hence,

$$\chi\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = \omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = p^{k_1-1}q^{k_2-1}r^{k_3-1} + 2.$$

Similarly, by Theorems 3.1, 3.2, 3.3, we get the proof for the remaining cases. Hence,

$$\chi\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right) = \omega\left(\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)\right),$$

for positive integers k_1, k_2, k_3. □

4 Conclusions

Lande and Khairnar [6] determined the adjacency spectrum of the generalized zero-divisor graph $\Gamma'(\mathbb{Z}_n)$ for $n = p^{\alpha}q^{\beta}$, where p, q are distinct primes and α, β are positive integers. This paper extend the study of adjacency spectrum of $\Gamma'(\mathbb{Z}_n)$ to $n = p^{k_1}q^{k_2}r^{k_3}$. Additionally, the clique number, stability number, diameter and girth are determined. Also, it is proved that Beck's conjecture holds for $\Gamma'\left(\mathbb{Z}_{p^{k_1}q^{k_2}r^{k_3}}\right)$.

4.1 Future Scope

The future research directions include:

1. Extending the study of adjacency spectrum of $\Gamma'(\mathbb{Z}_n)$ to $n = p_1p_2 \dots p_k$ for distinct primes $p_1, p_2, \dots, p_k$ and then, extending the results to any positive integer n.
2. Investigating other spectra viz. the Laplacian spectrum, the signless Laplacian spectrum, the normalized distance Laplacian spectrum, the Randić spectrum, the A_α spectrum, etc. of $\Gamma'(\mathbb{Z}_n)$ and exploring the topological indices.
3. Exploring the study of the spectrum of $\Gamma'(R)$ for non-commutative rings.

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